THE FIRST-ORDER DELTA-SIGMA MODULATOR

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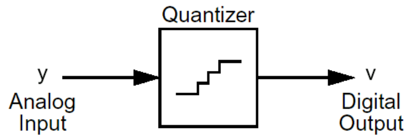
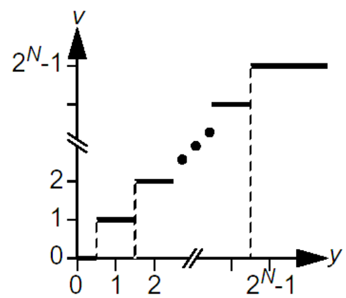
Outline

- Quantizers and quantization noise
- Binary quantization
- MOD1 as an ADC
- MOD1 as a DAC
- MOD1 linear model
- Simulation of MOD1
- MOD1 under DC excitation
- The effects of finite op-amp gain
- Decimation filters for MOD1

Quantizers and Quantization Noise (1)

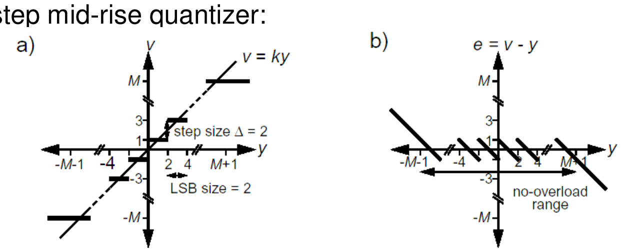
- Unipolar N-bit quantizer:



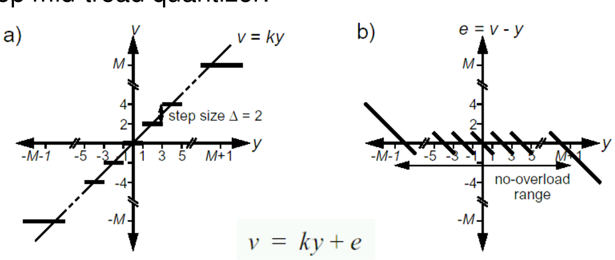


Quantizers and Quantization Noise (2)

- M-step mid-rise quantizer:



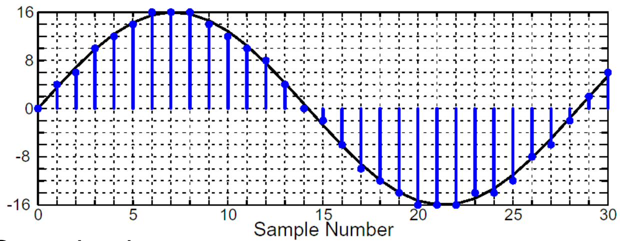
- M-step mid-tread quantizer:



$$v = ky + e$$

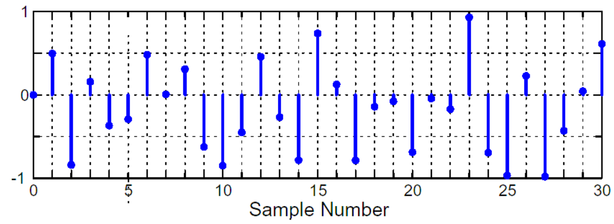
Quantizers and Quantization Noise (3)

- Sampled signal:



16-step quantization used

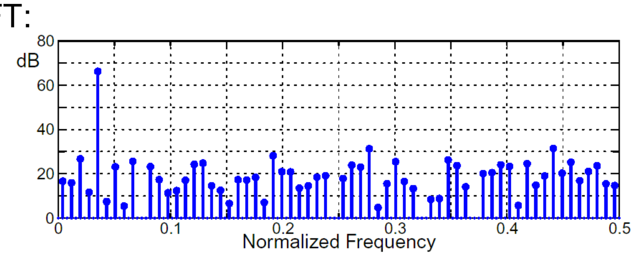
- Quantization error:



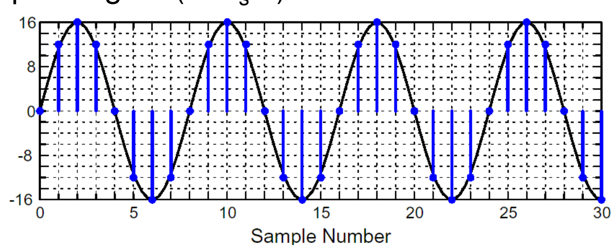
f/f_s irrational!

Quantizers and Quantization Noise (4)

- FFT:

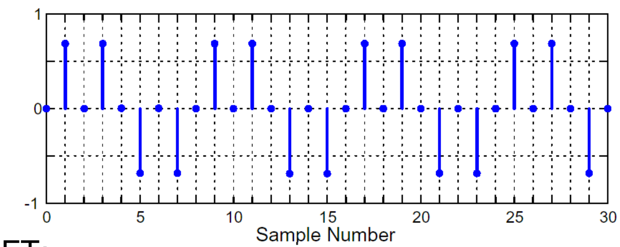


- Sampled signal ($f = f_s/8$):

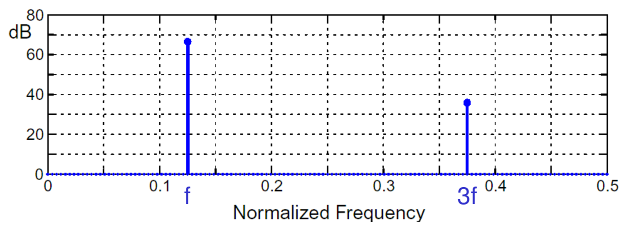


Binary Quantization (1)

- Quantization error:



- FFT:



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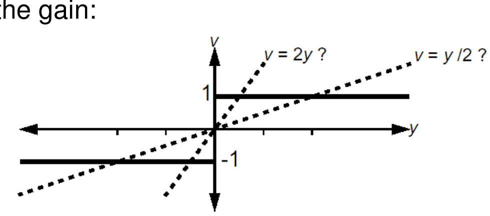
Binary Quantization (2)

- Modeling the gain:



- Minimize mean square error of *e*: $\sigma_e^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} e(n)^2$

$$\langle a, b \rangle \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} a(n)b(n) = E[ab]$$

$$\begin{aligned} \sigma_e^2 &= \langle e, e \rangle \\ &= \langle v - ky, v - ky \rangle \\ &= \langle v, v \rangle - 2k\langle v, y \rangle + k^2 \langle y, y \rangle . \end{aligned}$$

opt.: $k = \frac{\langle v, y \rangle}{\langle y, y \rangle} = \frac{E[|y|]}{E[y^2]}$

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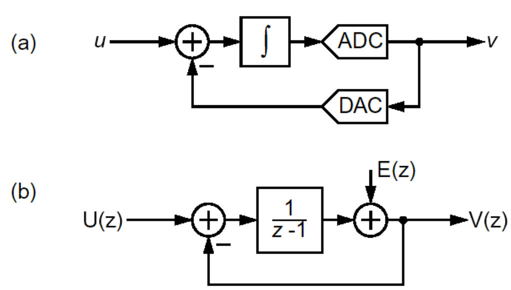
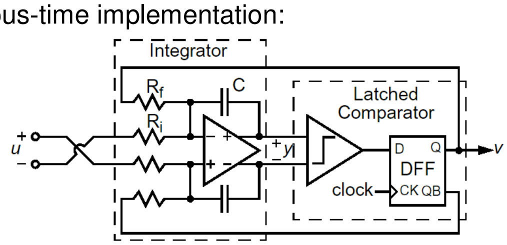
MOD1 as an ADC (1)

- Linear modeling:

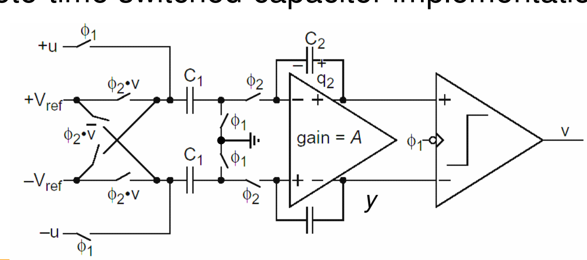


MOD1 as an ADC (2)

- Continuous-time implementation:

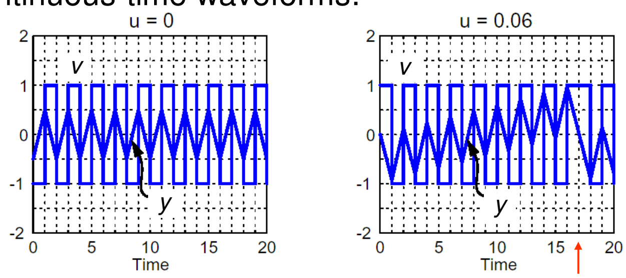


- Discrete-time switched-capacitor implementation:



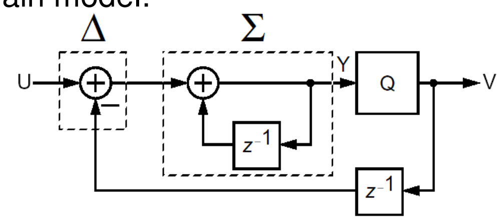
MOD1 as an ADC (3)

- Continuous-time waveforms:



change in v pattern

- Z-domain model:



MOD1 as an ADC (4)

- Stable operation:

$$v(n) = \text{sgn}[y(n)].$$

$$y(n) = y(n-1) + u(n) - v(n-1)$$

$$y(N) - y(0) = \sum_{n=0}^{N} [u(n) - v(n-1)].$$

If $y(n)$ is bounded,

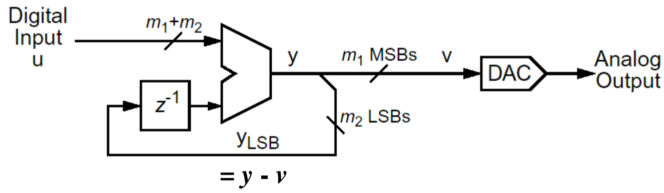
$$\lim_{N \to \infty} \frac{y(N) - y(0)}{N} = 0$$

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u(n) = \bar{v} \quad !$$

Perfectly accurate for $N \to \infty$.

MOD1 as a DAC

- Error feedback structure: → recycled error!



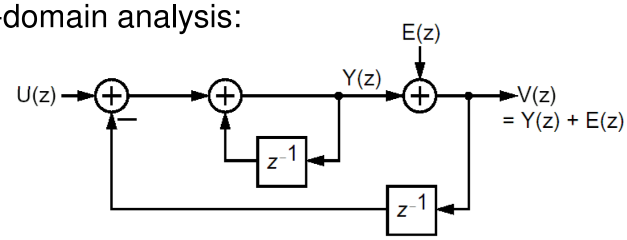
$$y(n) = u(n) + y_{LSB}(n-1)$$

$$y(n) = u(n) + y(n-1) - v(n-1)$$

Same as for ΔΣ loop → another option for DAC.
(For ADC, impractical!)

MOD1 Linear Model (1)

- Z-domain analysis:



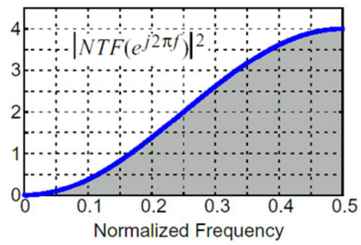
$$Y(z) = z^{-1}Y(z) + U(z) - z^{-1}V(z)$$

$$\begin{aligned} V(z) &= Y(z) + E(z) = z^{-1}Y(z) + U(z) - z^{-1}V(z) + E(z) \\ &= U(z) + E(z) - z^{-1}(V(z) - Y(z)) \\ &= U(z) + E(z) - z^{-1}E(z) \\ &= U(z) + (1 - z^{-1})E(z). \end{aligned}$$

$$V(z) = STF(z)U(z) + NTF(z)E(z)$$

MOD1 Linear Model (2)

- Frequency-domain analysis: $|NTF(e^{j2\pi f})|^2 = [2\sin(\pi f)]^2$



Mean square of q_o: $\sigma_{q_0}^2 = \int_0^{1/(2 \cdot OSR)} [2\pi f]^2 S_e(f) df = \frac{\pi^2}{9(OSR)^3}$ (for OSR >> 1)

(inband shaped quant. noise)

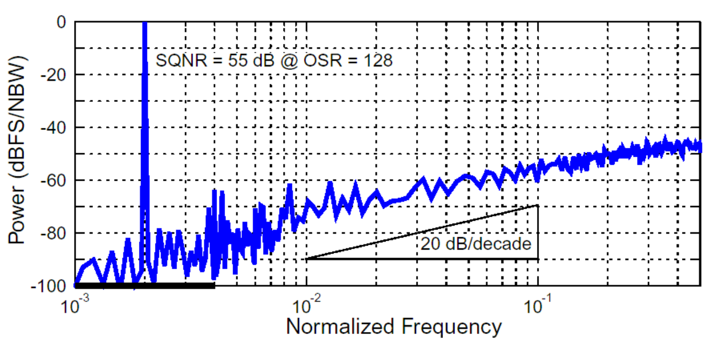
# of levels in Q

$$SQNR = \frac{\sigma_u^2}{\sigma_{q_0}^2} = \frac{9M^2(OSR)^3}{2\pi^2}$$

Signal-to-quantization noise ratio

Simulation of MOD1 (1)

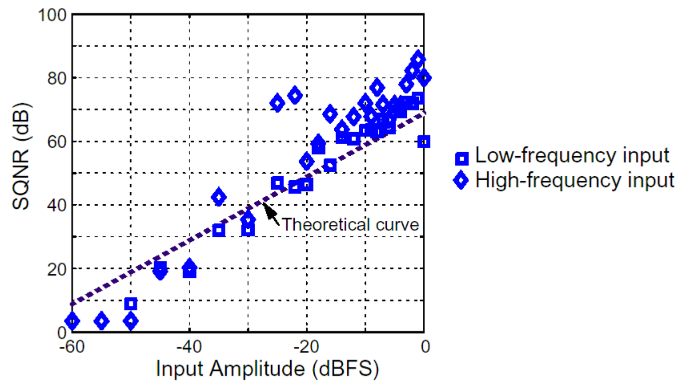
- Output spectrum for full-scale sine-wave input:



Looks ok, but SQNR 5 dB less than the formulaic.

Simulation of MOD1 (2)

- SQNRs for different frequencies:

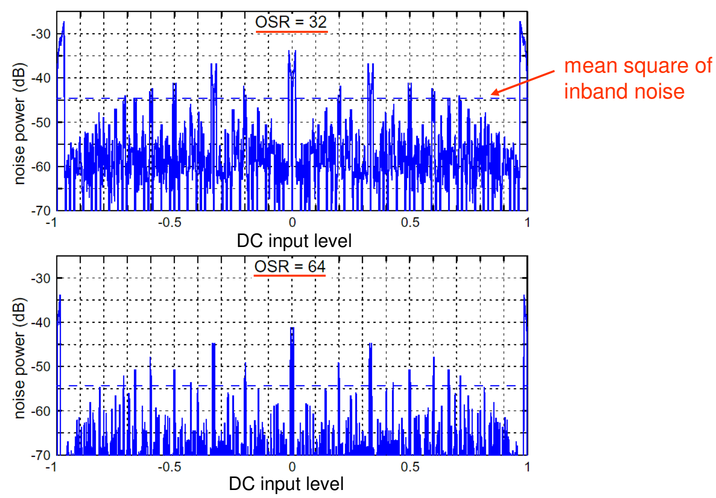


Sine wave inputs

Simulation of MOD1 (3)

- In-band quantization noise power:

$V_{ref} = \pm 1$



MOD1 Under DC Excitation (1)

- Idle tones:

$$y(n) = y(n-1) + u - v(n-1)$$

$$v(n) = \mathrm{sgn}(y(n)).$$

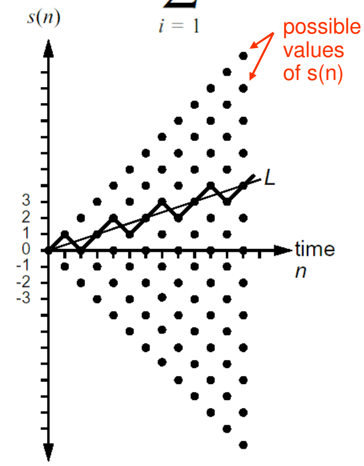
$$y(n) = y(n-1) + u - \mathrm{sgn}(y(n-1))$$

- u = y(0) = ½:

n	0	1	2	3	4
$y(n)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{1}{2}$
$v(n)$	1	1	-1	1	1

- For u = 0.01, tones at $k \cdot f_s/200$!
  k = 1, 2, …

$$s(n) = \sum_{i=1}^{n} v(i)$$



MOD1 Under DC Excitation (2)

- Let $u = a/b$, a and b odd integers, and $0 < a < b$. Also, let $|y(0)| < 1$. Then, the output has a period b samples. In each period, $v(n)$ will contain $(b+a)/2$ samples of +1, and $(b-a)/2$ samples of -1.
- If a or b is even, the period is $2b$, with $(a+b)$ +1s and $(b-a)$ -1s.

- If $v(n)$ has a period p, with n +1s and $p-m$ -1s, the average $\overline{v} = (2m - p)/p$. Hence, $u = \overline{v}$ is also rational. Thus, rational dc $u \Leftrightarrow$ periodic $v(n)$.
- Periodic $v(n)$: pattern noise, idle tone, limit cycle. Not instability!
- For $u = 1/100$, tones at $k \cdot f_s/200$, k = 1, 2, … some may be in the baseband. Often intolerable!

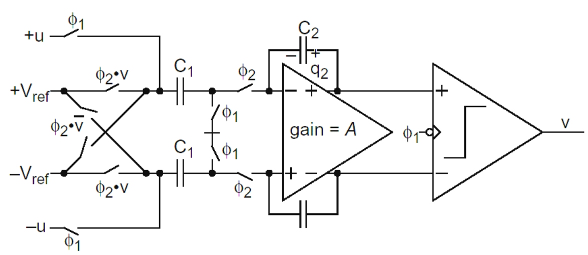
Stability of MOD1

- MOD1 always stable as long as |u| ≤ 1, and |y(0)| ≤ 2:

$$y(n) = \underbrace{\left[y(n-1) - \mathrm{sgn}\left(y(n-1)\right)\right]}_{\|[\ ]\| \le 1} + \underbrace{u(n)}_{\le 1} \quad \le 2$$

- If $u > 1$ (or $u < 1$), v will always be +1 (or -1) ⇒ y will increase (or decrease) indefinitely.
- If |u(n)| ≤ 1 but |y(0)| > 2, then |y(n)| will decrease to < 2. Output spectrum always a line spectrum for MOD1 with dc input (rational or not).

The Effects of Finite Op-Amp Gain (1)

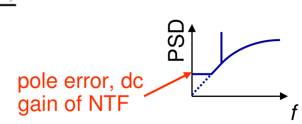


- Degraded noise shaping:

$$q_2(n) = q_2(n-1) + C_1\left(u(n) - v(n-1) - \frac{q_2(n)}{C_2(A+1)}\right)$$

$$Y(z) = p\frac{zU(z) - V(z)}{z-p}$$

$$NTF(z) = 1 - pz^{-1} \quad \to 1 - p = 1/A$$



The Effects of Finite Op-Amp Gain (2)

- Dead zones:

Ideally:

$$y(1) = y(0) + u - \text{sgn}(y(0)) = u - 1 < 0 \qquad v = -1$$
$$y(2) = u - 1 + u + 1 = 2u > 0 \qquad +1$$
$$y(3) = 2u + u - 1 = 3u - 1 < 0 \qquad -1$$

$$y(k) = \begin{cases} ku - 1 & \text{, if } k \text{ is odd} \\ ku & \text{, if } k \text{ is even} \end{cases}$$

for $u > 0$, eventually $ku > 1$ and two 1's occur.

For A < ∞: $y(n) = py(n-1) + u - \text{sgn}(y(n-1))$, $p = 1 - 1/A$

$$y(1) = y(0) + u - \text{sgn}(y(0)) = u - 1 < 0 \qquad v = -1$$
$$y(2) = pu - p + u + 1 = (1+p)u + (1-p) > 0 \qquad +1$$
$$y(3) = p(1+p)u + p(1-p) + u - 1 = (1 + p + p^2)u - (1 - p + p^2) < 0 \qquad -1$$
$$\ldots$$
$$y(k) = \sum_{i=0}^{k-1} p^i u + (-1)^k \sum_{i=0}^{k-1} (-p)^i$$

The Effects of Finite Op-Amp Gain (3)

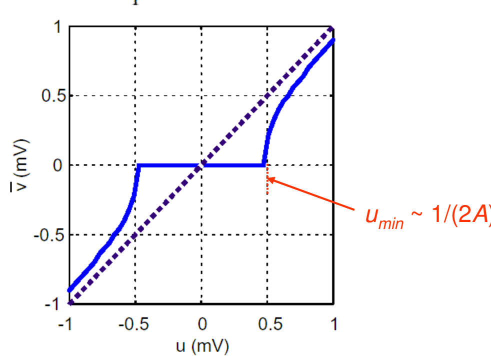
- For $\bar{v} > 0$,

(Two 1's occuring)

$$\frac{u}{1-p} > \frac{1}{1+p}$$

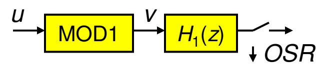
$$u > \frac{1-p}{1+p} = \frac{1/A}{2 - 1/A} \approx \frac{1}{2A}$$

For $A \approx 10^3$:



Decimation Filters for MOD1 (1)

- The sinc filter:



Averaging over N samples (running-average)

$$w(n) = \frac{1}{N}\sum_{i=0}^{N-1} v(n-i)$$

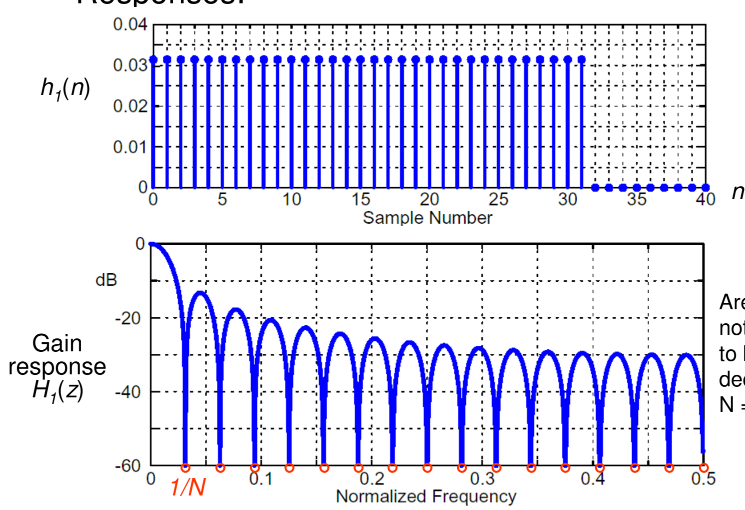
$$h_1(n) = \begin{cases} 1/N, & \text{if } (0 \le n \le N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$H_1(z) = \frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}}$$

$$H_1(e^{j2\pi f}) = \frac{\text{sinc}(Nf)}{\text{sinc}(f)} \qquad \text{sinc}(f) \triangleq \frac{\sin(\pi f)}{\pi f}$$

Decimation Filters for MOD1 (2)

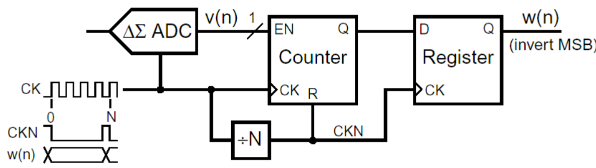
- Responses:



Areas around notches fold back to baseband after decimation if N = OSR.

Decimation Filters for MOD1 (3)

- Implementation:



$$Q_1(z) = H_1(z)NTF(z)E(z) = \frac{1}{N}(1 - z^{-N})E(z)$$

$$q_1(n) = \frac{1}{N}[e(n) - e(n-N)]$$

Assuming $e(n)$ and $e(n - N)$ are uncorrelated:

Inband noise after H_1:	Inband noise before H_1:
$\sigma^2_{q_1} = \frac{2e^2_{rms}}{N^2}$ ($\leftarrow \sigma_e^2$)	$\sigma^2_{q_0} = \frac{\pi^2\sigma^2_e}{3N^3}$
Total noise after H_1; Too much!	Total noise after ideal LPF; Much less than σ_{q1}^2!

Decimation Filters for MOD1 (4)

- The sinc² filter:

$$H_2(z) = \left[\frac{(1 - z^{-N})}{N(1 - z^{-1})}\right]^2$$

$$H_2(e^{j2\pi f}) = \left(\frac{\mathrm{sinc}(Nf)}{\mathrm{sinc}(f)}\right)^2$$

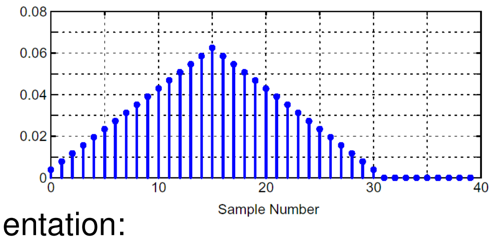
$$Q_2(z) = NTF(z)H_2(z)E(z) = \frac{1}{N^2}\frac{(1 - z^{-N})}{(1 - z^{-1})}(1 - z^{-N})E(z) = \frac{1}{N}H_1(z)[(1 - z^{-N})E(z)]$$

$$q_2(n) = \frac{1}{N^2}\sum_{i=0}^{N-1}[e(n-i) - e(n-N-i)]$$

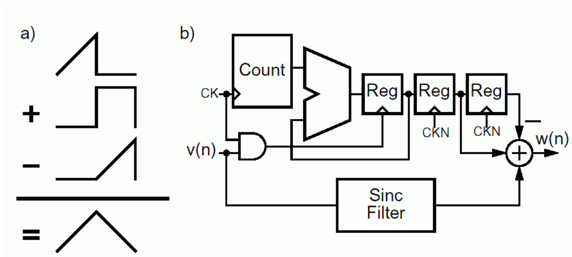
$$\sigma^2_{q_2} = \frac{2N\sigma^2_e}{N^4} = \frac{2\sigma^2_e}{N^3}$$

Decimation Filters for MOD1 (5)

- Response:



- Implementation:



References

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